Adic representation of a product transformation

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Abstract

These notes are an attempt to explain how to construct an adic representation of $T \times S$ when S is an adic transformation and T a transformation with a finite generator.

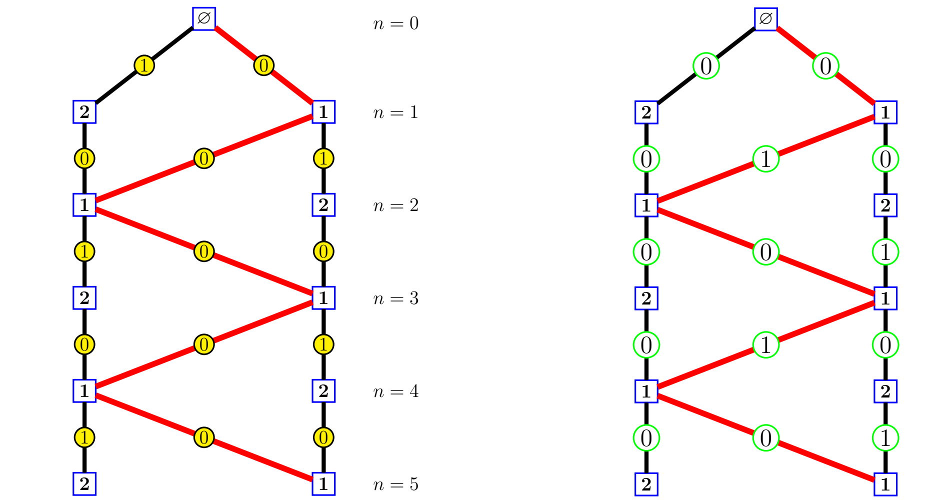
1 Adic transformation S

Consider a Bratteli graph, for example the golden graph shown on Figure 1. The dimension of the vertex 1 at level n is f_n and the dimension of the vertex 2 at level n is f_{n-1}, where (f_n) is the Fibonacci sequence.

An infinite path γ starting at the root vertex $\varnothing$ is a sequence of arcs $\gamma = (\gamma_0, \gamma_1, \ldots)$ where γ_n connects a vertex at level n to a vertex at level $n+1$. We denote by Γ the set of infinite paths.

The labels on the arcs shown on Figure 1(a), provide, for each vertex v_n at a level n, an ordering of the arcs between v_n and the vertices connected to v_n at level $n+1$. These labels allow to identify an infinite path (starting at the root vertex $\varnothing$).

The labels shown on Figure 1(b) are obtained by considering the other direction: they provide, for each vertex v_{n+1} at a level $n+1$, an ordering of the arcs between v_{n+1} and the vertices at level n connected to v_{n+1}. I call them the *order labels*. For a path γ passing through vertex v_n at level n and vertex v_{n+1} at level $n+1$, we denote by $\epsilon_n(\gamma)$ the order label of the arc of γ connecting v_n and v_{n+1}.



(a) The labels on the arcs identify the paths

(b) Order labels ϵ_n on the arcs

Figure 1: Golden graph

The tree on Figure 2 shows all the paths from the vertex $v_5 = 1$ to the root vertex (we drop level $n = 0$ because it is useless, for there is an unique arc between each vertex at level $n = 1$ and the root vertex at level $n = 0$).

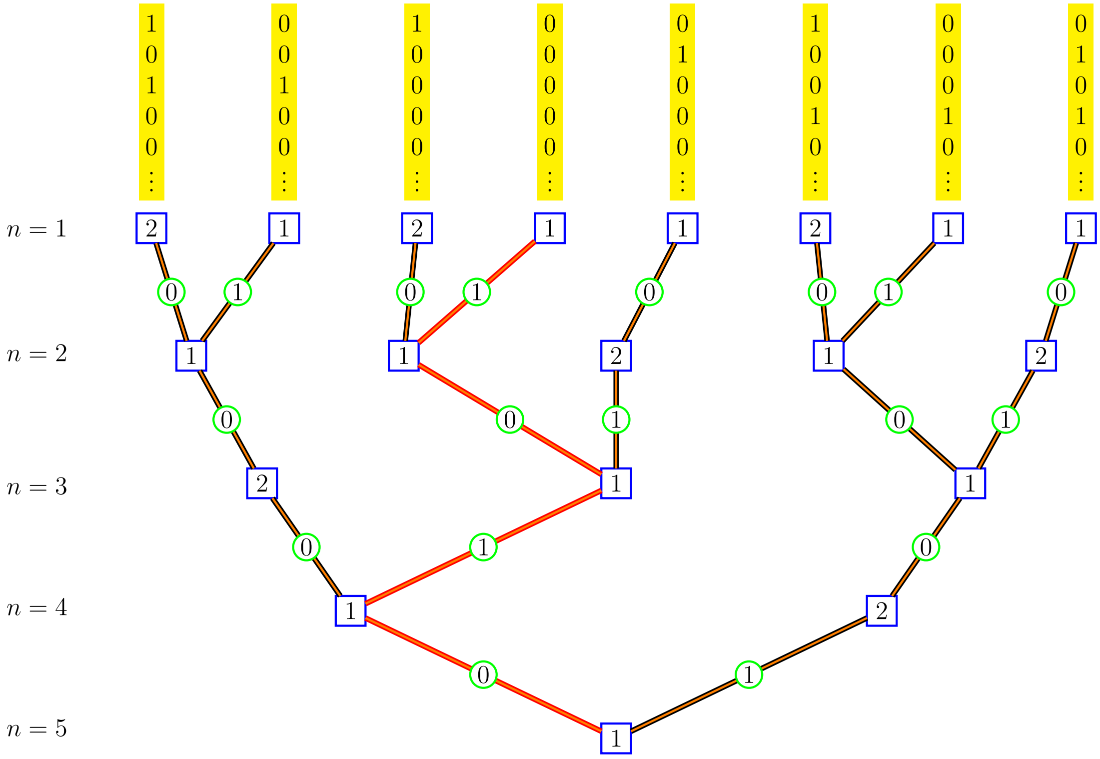


Figure 2: Paths from the vertex 1 at level $n = 5$

Consider a set of paths passing through the vertex $v_5 = 1$ and with the same arcs at levels $n \geq 5$. The tree on Figure 2 shows how these paths are ordered: from the left to the right on the figure. The *adic transformation* S sends a path to its successor for this order. The sequence $(\epsilon_0(S\gamma), \dots, \epsilon_4(S\gamma))$ of order labels of $S\gamma$ is the successor of the sequence $(\epsilon_0(\gamma), \dots, \epsilon_4(\gamma))$ of order labels of γ for the anti-lexicographical order.

2 Adic representation of a product $T \times S$

Consider an automorphism T with a finite generating partition P. For our illustration, we take the rotation with angle $\theta = \sqrt{2} - 1$ and the partition $P = \Big\{[0, \theta[, [\theta, 1[\Big\}$.

The automorphism T is isomorphic to the shift on a subset $\mathcal{W}$ of $\{a, b\}^{\mathbb{Z}}$, namely, the set of all sequences obtained by coding the orbits $\{T^n x\}_{n \in \mathbb{Z}}$ by a or b according to whether it is in $[0, \theta[$ or $[\theta, 1[$. The finite subwords of a word in $\mathcal{W}$ are the subwords of the Sturmian word $bababbababbabababbababba\dots$.

The Bratteli graph shown on Figure 3 is obtained as follows. At level n, we write all the possible words of length f_{n-1} and f_n, the two dimensions of the vertices at level n of the golden graph. These words are the vertices of the graph. Then we connect each word at level $n+1$ to its subwords at level n, by respecting the structure of the golden graph (see Figure 4).

This graph is a representation of the set $\mathcal{W} \times \Gamma$. For example, the path shown in red on Figure 3 corresponds to $w = \dots abbab \dots \in \mathcal{W}$ and $\gamma = (0, 0, 0, 0, \dots) \in \Gamma$.

Figure 4(a) helps to see the action of the adic transformation on this graph. It is clear that the adic transformation sends γ to $S\gamma$, and it sends $\dots abbab \dots$ to $\dots abbab \dots$. Hence it acts like $T \times S$.

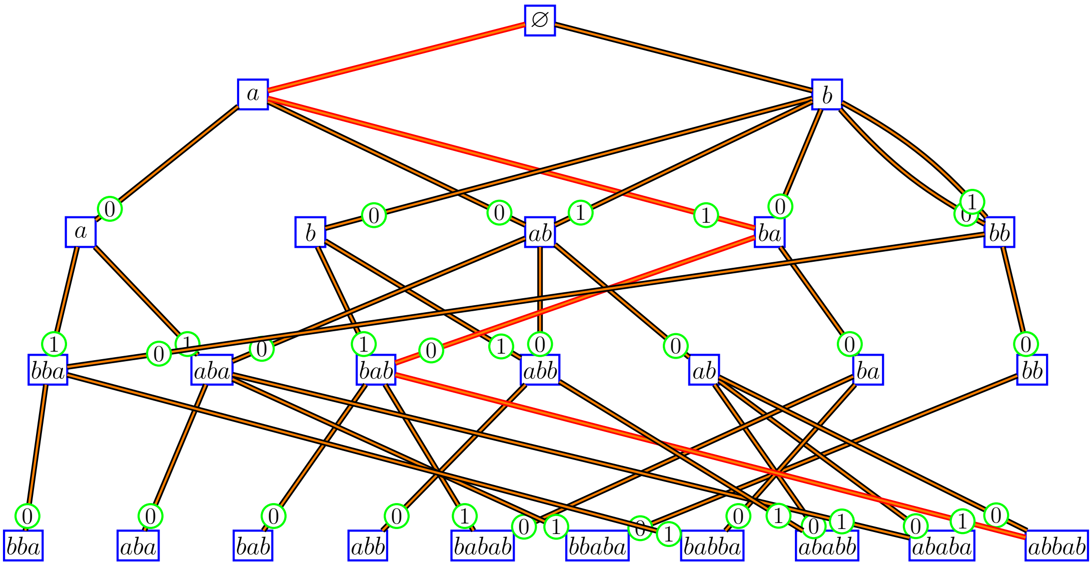
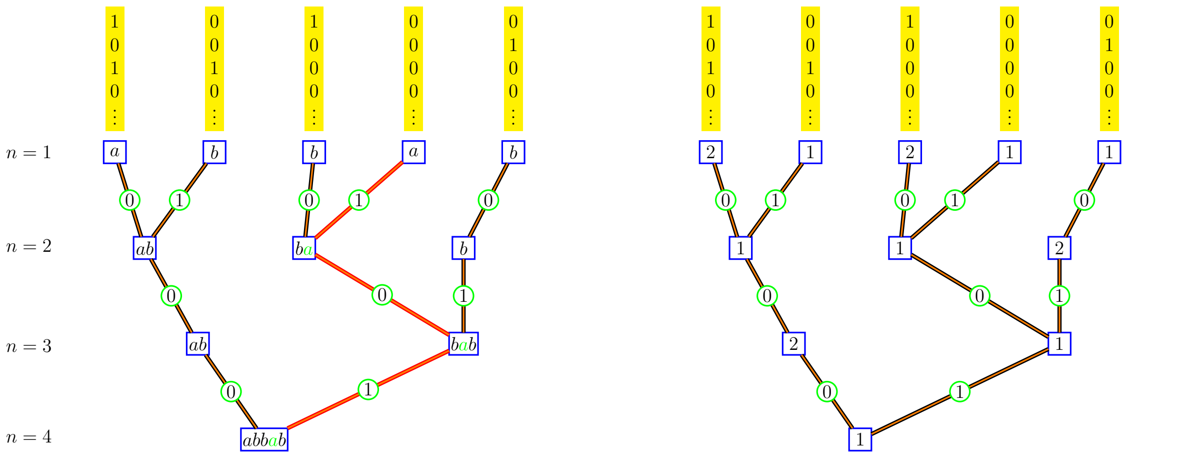


Figure 3



(a) paths from the vertex $v_4 = abbab$

(b) (golden graph) paths from the vertex $v_4 = 1$

Figure 4